

# Theory of Hall Effect in a Two-Dimensional Electron System

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Hall conductivity  $\sigma_{xy}$  is studied in various approximations. Characteristics of  $\sigma_{xy}$  are obtained for the case of both short- and long-ranged scatterers in the self-consistent Born approximation, which is the simplest one free from the difficulty of divergence. In case of short-ranged scatterers, a relation is shown to hold between  $\sigma_{xy}$  and  $\sigma_{xx}$  within this approximation, if one uses a relaxation time under magnetic fields. Under strong magnetic fields, effects of higher Born scattering become important in low-lying Landau levels. They depend on the sign of scatterers and strongly on their concentrations. Effects of simultaneous scattering from many scatterers are calculated to the lowest order.

## §1. Introduction

When a strong magnetic field is applied perpendicularly to a two-dimensional electron system such as inversion layers on semi-conductor surfaces, the energy spectrum becomes discrete because of the complete quantization of the orbital motion. Such a singular system provides an ideal tool for studying the quantum transport phenomena. In a series of previous papers,<sup>1-4)</sup> which are referred to as I, II, III and IV in what follows, the transverse conductivity  $\sigma_{xx}$  was studied systematically. The present paper is concerned with another important quantity—the Hall conductivity  $\sigma_{xy}$ .

In order to see characteristics of  $\sigma_{xy}$ , we first employ the self-consistent Born approximation (SCBA) which is the simplest one free from the difficulty of divergence caused by the singular nature of our system.<sup>1)</sup> One takes into account effects of scattering from an impurity in the lowest Born approximation, while the collision broadening is included in a self-consistent manner.

According to a simple phenomenological argument, the Hall conductivity  $\sigma_{xy}$  becomes

$$\begin{aligned}\sigma_{xy} &= -\frac{nec}{H} \frac{\omega_c^2 \tau^2}{1 + \omega_c^2 \tau^2} \\ &= -\frac{nec}{H} + \frac{1}{\omega_c \tau} \frac{ne^2 \tau}{m} \frac{1}{1 + \omega_c^2 \tau^2},\end{aligned}\quad (1.1)$$

where  $n$  is the total number of electrons in a unit area,  $\omega_c$  is the cyclotron frequency given

by  $\omega_c = eH/mc$ , and  $\tau$  is the relaxation time. The above equation can be written as

$$\sigma_{xy} = -\frac{nec}{H} + \Delta\sigma_{xy}, \quad (1.2)$$

with

$$\Delta\sigma_{xy} = \frac{1}{\omega_c \tau} \sigma_{xx}. \quad (1.3)$$

In the SCBA such kind of relationship between  $\Delta\sigma_{xy}$  and  $\sigma_{xx}$  holds in case of short-ranged scatterers if a relaxation time under the magnetic field is used in eq. (1.3). In contrast to the case of  $\sigma_{xx}$ , however, contributions from higher order approximations are relatively important and can be crucial especially under extremely strong magnetic fields. Those higher order corrections are also investigated.

In §2, starting from the center migration theory of Kubo *et al.*<sup>5,6)</sup> and using the technique of Green's function, we obtain necessary equations of  $\Delta\sigma_{xy}$ . In §3,  $\sigma_{xy}$  is explicitly calculated in the SCBA under strong magnetic fields and characteristics in case of short- and long-ranged scatterers are obtained. Assuming scatterers with a short-ranged potential, we calculate the oscillatory  $\sigma_{xy}$  under magnetic fields of arbitrary strength and show that eq. (1.3) holds. In the first part of §4, we investigate effects of higher Born scattering and show that they are important in low-lying Landau levels especially when the concentration of scatterers is not so large. Effects of simultaneous scattering from more than two scatterers are also investigated and are shown to be relatively important in low-lying Landau levels. Some discussions on the obtained results are given in §5.

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## §2. Hall Conductivity $\sigma_{xy}$

We consider a two-dimensional system described by the Hamiltonian

$$H = \frac{1}{2m} \left( \mathbf{p} + \frac{e}{c} \mathbf{A} \right)^2 + \sum_{\mu} \sum_i v^{(\mu)}(\mathbf{r} - \mathbf{R}_i, Z_i), \quad (2.1)$$

where bold-faced letters represent two-dimensional vectors in the  $xy$ -plane,  $\mathbf{A} = (-Hy/2, Hx/2)$ , and  $v^{(\mu)}(\mathbf{r} - \mathbf{R}_i, Z_i)$  is effective two-dimensional potential of a scatterer located at  $(\mathbf{R}_i, Z_i)$ . The potential is assumed to be cylindrically symmetric for simplicity. It can be written in a mixed representation<sup>1,5)</sup>

$$H = \sum_{NX} E_N a_{NX}^+ a_{NX} + \sum_{\mu} \sum_i \sum_{NX} \sum_{N'X'} \sum_m 2\pi l^2 \varphi_{N+mX}^*(\mathbf{R}_i) \times \varphi_{N'+mX'}(\mathbf{R}_i) (Nm|v^{(\mu)}(Z)|N'm) a_{NX}^+ a_{N'X'}, \quad (2.2)$$

where  $E_N = (N+1/2)\hbar\omega_c$ ,  $l^2 = \hbar^2/eH$ ,  $a_{NX}^+$  is the creation operator of an electron in the  $N$ -th Landau level having a center coordinate  $X$ ,  $\phi_{NX}(\mathbf{r})$  is its wave function, and  $(Nm|v^{(\mu)}(Z)|N'm)$  is a matrix element of a potential between states with an angular momentum  $m$  around the position of a scatterer.

According to the center migration theory of Kubo *et al.*<sup>5,6)</sup> one has

$$\Delta\sigma_{xy} = \frac{e^2\hbar}{i\pi L^2} \int f(E) dE \left\langle \text{Tr} \dot{X} \left( \frac{\partial}{\partial E} \text{Re} \frac{1}{E-H+i0} \right) \dot{Y} \text{Im} \frac{1}{E-H+i0} - (\dot{X} \leftrightarrow \dot{Y}) \right\rangle, \quad (2.3)$$

where  $L^2$  is the area of the system,  $f(E)$  is the Fermi distribution function, a trace should be taken over one-electron states,  $\langle \dots \rangle$  means an average over all configurations of scatterers, and

$$\begin{aligned} \begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} &= \frac{l}{\hbar} \sum_{\mu} \sum_i \begin{pmatrix} l \frac{\partial}{\partial y} \\ -l \frac{\partial}{\partial x} \end{pmatrix} v^{(\mu)}(\mathbf{r} - \mathbf{R}_i, Z_i) \\ &= -\frac{l}{\hbar} \sum_{\mu} \sum_i \sum_{NX} \sum_{N'X'} \sum_m \sum_{\pm} \begin{pmatrix} (Nm | l \frac{\partial v^{(\mu)}(Z)}{\partial y} | N'm \pm 1) \\ - (Nm | l \frac{\partial v^{(\mu)}(Z)}{\partial x} | N'm \pm 1) \end{pmatrix} \\ &\quad \times 2\pi l^2 \varphi_{N+mX}^*(\mathbf{R}_i) \varphi_{N'+m\pm 1X'}(\mathbf{R}_i) a_{NX}^+ a_{N'X'}. \end{aligned} \quad (2.4)$$

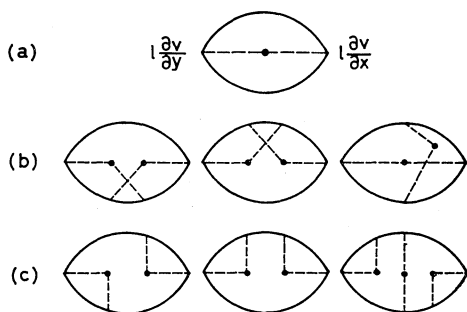


Fig. 1. Examples of diagrams of  $\Delta\sigma_{xy}^{(1)}$  and  $\Delta\sigma_{xy}^{(2)}$ . (a)  $\Delta\sigma_{xy}^{(1)}$  in the SCBA, which becomes identically zero, (b)  $\Delta\sigma_{xy}^{(1)}$  in the lowest double-site approximation (see §4), and (c) examples of  $\Delta\sigma_{xy}^{(2)}$  in the SCBA.

We divide  $\Delta\sigma_{xy}$  into two parts:  $\Delta\sigma_{xy}^{(1)}$  and  $\Delta\sigma_{xy}^{(2)}$ . The former  $\Delta\sigma_{xy}^{(1)}$  represents contributions

from those diagrams which can not be cut into two parts by cutting two internal electron lines, and the latter  $\Delta\sigma_{xy}^{(2)}$  contributions from other diagrams. Examples are shown in Fig. 1. As will be shown in the following,  $\Delta\sigma_{xy}^{(1)}$  and  $\Delta\sigma_{xy}^{(2)}$  are different in nature.

Introduce a quantity  $\xi_{N+1,N}^x(E', E)$  or  $\xi_{N+1,N}^y(E', E)$ , which we call  $\xi$ -part. Each  $\xi$ -part is

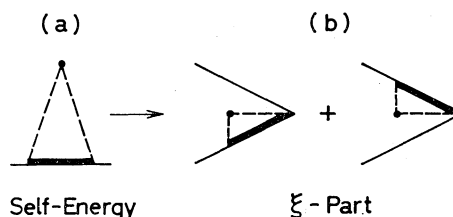


Fig. 2. Self-energy in the SCBA and corresponding  $\xi$ -part.

obtained graphically from self-energy diagrams by replacing one of matrix elements of the potential  $v^{(\mu)}(\mathbf{r}, Z)$  by the corresponding  $\partial v^{(\mu)}(\mathbf{r}, Z)/\partial x$  or  $\partial v^{(\mu)}(\mathbf{r}, Z)/\partial y$  as is shown in Fig. 2. From graphical consideration one gets

$$\Delta\sigma_{XY}^{(2)} = \frac{e^2}{8\pi^2\hbar} \int f(E) dE \sum_{N_1} \sum_{\pm} \lim_{E' \rightarrow E} \text{Im}_{(E)} \text{Re}_{(E')} \frac{\partial}{\partial E'} [\xi_{N_1 \pm 1 N_1}^y(E, E') \times G_{N_1}(E) G_{N_1 \pm 1}(E') \Xi_{N_1 \pm 1 N_1}^x(E', E) - (x \leftrightarrow y)] , \quad (2.5)$$

with

$$\Xi_{N_1 \pm 1 N_1}^x(E', E) = \xi_{N_1 \pm 1 N_1}^x(E', E) + \sum_{N_2} \gamma_{N_1 \pm 1 N_1, N_2 \pm 1 N_2}(E', E) G_{N_2 \pm 1}(E') G_{N_2}(E) \Xi_{N_2 \pm 1 N_2}^x(E', E) , \quad (2.6)$$

where  $\gamma_{N_1 \pm 1 N_1, N_2 \pm 1 N_2}$  is the proper vertex part defined in I and  $G_N(E)$  is average Green's function

$$G_N(E) = \left\langle \left( 0 \left| a_{NX} \frac{1}{E-H} a_{NX}^\dagger \right| 0 \right) \right\rangle . \quad (2.7)$$

Making use of

$$\left( Nm \pm 1 \left| l \frac{\partial v^{(\mu)}(Z)}{\partial y} \right| N'm \right) = \pm i \left( Nm \pm 1 \left| l \frac{\partial v^{(\mu)}(Z)}{\partial x} \right| N'm \right) , \quad (2.8)$$

one can show that

$$\xi_{N \pm 1 N}^y(E', E) = \mp i \xi_{N \pm 1 N}^x(E', E) . \quad (2.9)$$

Further, as is easily seen, one has

$$\left. \begin{aligned} \xi_{N \pm 1 N}^x(E', E) &= \bar{\xi}_{N \pm 1 N}^x(E, E') , \\ \xi_{N \pm 1 N}^x(E, E) &= \xi_{N \pm 1 N}^y(E, E) = 0 , \end{aligned} \right\} \quad (2.10)$$

where  $\bar{\xi}$  can be obtained from  $\xi$  by taking the complex conjugate of matrix elements of potentials appearing in  $\xi$ . By the use of above relations, eq. (2.5) becomes

$$\Delta\sigma_{XY}^{(2)} = \frac{e^2}{4\pi^2\hbar} \int \left( -\frac{\partial f}{\partial E} \right) dE \sum_{N_1} \sum_{\pm} (\pm i) \xi_{N_1 N_1 \pm 1}^x(E+i0, E-i0) \times G_{N_1}(E+i0) G_{N_1 \pm 1}(E-i0) \Xi_{N_1 \pm 1 N_1}^x(E-i0, E+i0) . \quad (2.11)$$

Therefore,  $\Delta\sigma_{XY}^{(2)}$  is determined only from quantities at the Fermi energy at zero temperature. Next let us consider the case under strong magnetic fields and assume that the Fermi level lies in the  $N$ -th Landau level. In this case, one has to retain terms with  $N_1=N$  and  $N+1$  in eq. (2.11), and further one has  $\Xi_{N \pm 1 N}^x = \xi_{N \pm 1 N}^x$  and  $G_{N \pm 1} = \mp (\hbar\omega_c)^{-1}$ . Therefore, eq. (2.11) is reduced to

$$\Delta\sigma_{XY}^{(2)} = \frac{e^2}{2\pi^2\hbar} \int \left( -\frac{\partial f}{\partial E} \right) dE \frac{1}{\hbar\omega_c} \sum_{\pm} \xi_{N \pm 1 N}^x(E-i0, E+i0) \bar{\xi}_{N \pm 1 N}^x(E-i0, E+i0) \text{Im} G_N(E+i0) . \quad (2.12)$$

### § 3. Self-Consistent Born Approximation (SCBA)

#### 3.1 Case of strong magnetic fields

In the SCBA,  $\Delta\sigma_{XY}^{(1)}$  is given by the diagram shown in Fig. 1(a) and becomes

$$\Delta\sigma_{XY}^{(1)} = \frac{e^2}{\pi^2\hbar} \int f(E) dE 2\pi l^2 \sum_{\mu} \int dZ N_i^{(\mu)}(Z) \sum_{N_1} \sum_{N_2} \sum_m \sum_{\pm} \frac{1}{2i} \left[ \left( N_1 m \left| l \frac{\partial v^{(\mu)}(Z)}{\partial y} \right| N_2 m \pm 1 \right) \times \left( N_2 m \pm 1 \left| l \frac{\partial v^{(\mu)}(Z)}{\partial x} \right| N_1 m \right) - (x \leftrightarrow y) \right] \text{Im} G_{N_1}(E+i0) \frac{\partial}{\partial E} \text{Re} G_{N_2}(E+i0) , \quad (3.1)$$

where  $N_i^{(\mu)}(Z)$  is the concentration of scatterers in a unit volume. With the aid of eqs. (2.9) of I and (2.7) of III, one gets

$$\begin{aligned} & \sum_m \sum_{\pm} \left( N_1 m \left| l \frac{\partial v^{(\mu)}(Z)}{\partial y} \right| N_2 m \pm 1 \right) \left( N_2 m \pm 1 \left| l \frac{\partial v^{(\mu)}(Z)}{\partial x} \right| N_1 m \right) \\ &= \int \frac{d^2 r}{2\pi l^2} \int \frac{d^2 r'}{2\pi l^2} l \frac{\partial v^{(\mu)}(\mathbf{r}, Z)}{\partial y} l \frac{\partial v^{(\mu)}(\mathbf{r}', Z)}{\partial x'} J_{N_2 N_2}(\mathbf{r} - \mathbf{r}') J_{N_1 N_1}(\mathbf{r} - \mathbf{r}') , \end{aligned} \quad (3.2)$$

which becomes identically zero because of the cylindrical symmetry of the potential. Therefore,  $\Delta\sigma_{XY}^{(1)}$  vanishes in the SCBA.

Next let us consider  $\Delta\sigma_{XY}^{(2)}$ . As the  $\xi$ -part one should include those diagrams shown in Fig. 2(b).

$$\begin{aligned} \xi_{N \pm 1 N}^x(E', E) &= 2\pi l^2 \sum_{\mu} \int dZ N_i^{(\mu)}(Z) \sum_{N'} \sum_m \left\{ \left( N \pm 1 m \mp 1 \left| l \frac{\partial v^{(\mu)}(Z)}{\partial x} \right| N' m \right) (N' m | v^{(\mu)}(Z) | Nm) \right. \\ &\quad \times G_{N'}(E) + (N \pm 1 m | v^{(\mu)}(Z) | N' m) \left( N' m \left| l \frac{\partial v^{(\mu)}(Z)}{\partial x} \right| Nm \pm 1 \right) G_{N'}(E') \left. \right\} . \end{aligned} \quad (3.3)$$

Again with the aid of (2.9) of I and (2.7) of III, one has

$$\begin{aligned} & \sum_m (N \pm 1 m | v^{(\mu)}(Z) | N' m) \left( N' m \left| l \frac{\partial v^{(\mu)}(Z)}{\partial x} \right| Nm \pm 1 \right) \\ &= - \sum_m \left( N \pm 1 m \mp 1 \left| l \frac{\partial v^{(\mu)}(Z)}{\partial x} \right| N' m \right) (N' m | v^{(\mu)}(Z) | Nm) \\ &= - \int \frac{d^2 r}{2\pi l^2} \int \frac{d^2 r'}{2\pi l^2} l \frac{\partial v^{(\mu)}(\mathbf{r}, Z)}{\partial x} v^{(\mu)}(\mathbf{r}', Z) J_{N N \pm 1}(\mathbf{r} - \mathbf{r}') J_{N' N'}(\mathbf{r} - \mathbf{r}') \exp[i\theta(\mathbf{r}, \mathbf{r}')] . \end{aligned} \quad (3.4)$$

Therefore, one sees that the  $\xi$ -part (3.3) satisfies eq. (2.10). Under strong magnetic fields, it becomes

$$\xi_{N \pm 1 N}^x(E - i0, E + i0) = \frac{1}{4} (\Gamma_N^{\pm})^2 [-2i \operatorname{Im} G_N(E + i0)] , \quad (3.5)$$

where

$$(\Gamma_N^{\pm})^2 = 4 \cdot 2\pi l^2 \sum_{\mu} \int dZ N_i^{(\mu)}(Z) \sum_m (N \mp 1 m | v^{(\mu)}(Z) | Nm) \left( Nm \left| l \frac{\partial v^{(\mu)}(Z)}{\partial x} \right| Nm \pm 1 \right) . \quad (3.6)$$

Substitution of eq. (3.5) into eq. (2.12) yields

$$\Delta\sigma_{XY}^{(2)} = \frac{e^2}{\pi^2 \hbar} \int \left( - \frac{\partial f}{\partial E} \right) dE \frac{\Gamma_N}{\hbar \omega_c} \left( \frac{\Gamma_N^{xy}}{\Gamma_N} \right)^4 \left[ 1 - \left( \frac{E - E_N}{\Gamma_N} \right)^2 \right]^{3/2} , \quad (3.7)$$

with

$$(\Gamma_N^{xy})^4 = \sum_{\pm} |\Gamma_N^{\pm}|^4 , \quad (3.8)$$

$$\Gamma_N^2 = 4 \cdot 2\pi l^2 \sum_{\mu} \int dZ N_i^{(\mu)}(Z) \sum_m |(Nm | v^{(\mu)}(Z) | Nm)|^2 , \quad (3.9)$$

where use has been made of eq. (3.5) of I.

When scatterers are of sufficiently short range, one replaces the potential by a  $\delta$ -potential,

$$v^{(\mu)}(\mathbf{r}, Z) = V^{(\mu)}(Z) \delta^{(2)}(\mathbf{r}) . \quad (3.10)$$

Then

$$\left( N0 \left| l \frac{\partial v^{(\mu)}(Z)}{\partial x} \right| N' \pm 1 \right) = \mp \frac{V^{(\mu)}(Z)}{2\pi l^2} \sqrt{\frac{N' + 1/2 \pm 1/2}{2}} , \quad (3.11)$$

$$(N0 | v^{(\mu)}(Z) | N'0) = \frac{V^{(\mu)}(Z)}{2\pi l^2} , \quad (3.12)$$

and other matrix elements are zero. One gets

$$\Delta\sigma_{xy}^{(2)} = \int \left( -\frac{\partial f}{\partial E} \right) dE \frac{e^2}{\pi^2 \hbar} \frac{\Gamma}{\hbar \omega_c} \left( N + \frac{1}{2} \right) \left[ 1 - \left( \frac{E - E_N}{\Gamma} \right)^2 \right]^{3/2}, \quad (3.13)$$

and

$$(\Gamma_N^{xy})^4 = \left( N + \frac{1}{2} \right) \Gamma^4, \quad (3.14)$$

where  $\Gamma^2$  can be expressed in terms of the relaxation time  $\tau_f$  obtained in the Born approximation by assuming the same scatterers in the absence of magnetic fields.

$$\Gamma^2 = 4 \sum_{\mu} \int dZ N_i^{(\mu)}(Z) \frac{|V^{(\mu)}(Z)|^2}{2\pi l^2} = \frac{2}{\pi} \hbar \omega_c \frac{\hbar}{\tau_f}. \quad (3.15)$$

When scatterers are of sufficiently slowly-varying type, eq. (3.6) becomes

$$\begin{aligned} (\Gamma_N^{\pm})^2 &= 4 \cdot 2\pi l^2 \sum_{\mu} \int dZ N_i^{(\mu)}(Z) \int \frac{d^2 r}{2\pi l^2} \int \frac{d^2 r'}{2\pi l^2} l \frac{\partial v^{(\mu)}(\mathbf{r}, Z)}{\partial x} v^{(\mu)}(\mathbf{r} + \mathbf{R}) \\ &\quad \times J_{NN \pm 1}(\mathbf{R}) J_{NN}(\mathbf{R}) \exp(\mp i\theta(\mathbf{R})) \\ &\simeq -4 \cdot 2\pi l^2 \sum_{\mu} \int dZ N_i^{(\mu)}(Z) \int \frac{d^2 r}{2\pi l^2} \left[ l \frac{\partial v^{(\mu)}(\mathbf{r}, Z)}{\partial x} \right]^2 \sqrt{\frac{N+1/2 \pm 1/2}{2}}, \end{aligned} \quad (3.16)$$

where  $\theta(\mathbf{R}) = \tan^{-1} R_y/R_x$ . It can be written as

$$(\Gamma_N^{\pm})^2 = -2 \sqrt{\frac{N+1/2 \pm 1/2}{2}} \langle (l \nabla V(\mathbf{r}))^2 \rangle, \quad (3.17)$$

where  $V(\mathbf{r})$  is the local potential energy. Therefore, one has

$$(\Gamma_N^{xy})^4 = 4 \left( N + \frac{1}{2} \right) \langle (l \nabla V(\mathbf{r}))^2 \rangle^2, \quad (3.18)$$

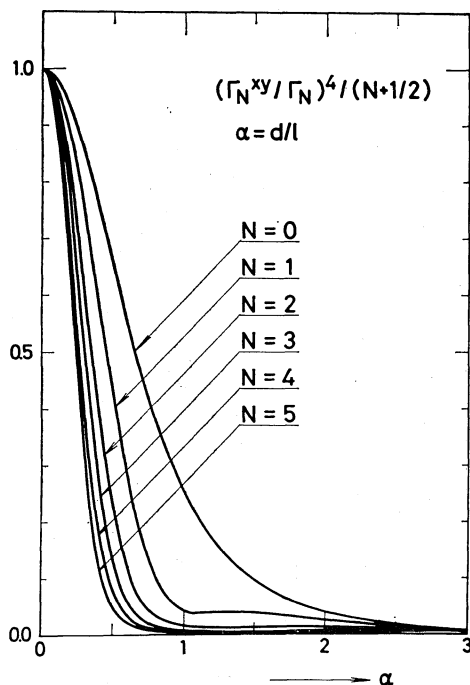


Fig. 3.  $(\Gamma_N^{xy}/\Gamma_N)^4/(N+1/2)$  as a function of  $\alpha = d/l$ .

and  $\Delta\sigma_{xy}$  decreases in proportion to  $(l/d)^4$ , where  $d$  is of the order of the range of scatterers.

In order to see such range dependence explicitly, we assume scatterers with a Gaussian potential

$$v^{(\mu)}(\mathbf{r}, Z) = \frac{V^{(\mu)}(Z)}{\pi d^2} \exp\left(-\frac{\mathbf{r}^2}{d^2}\right). \quad (3.19)$$

After a little manipulation, one gets

$$(\Gamma_N^{\pm})^2 = \Gamma^2 h_N^{\pm}(\alpha),$$

with

$$\begin{aligned} h_N^{\pm}(\alpha) &= \pm \frac{1}{\sqrt{2N+1 \pm 1}} \\ &\times \int_0^{\infty} dx x L_N^0(\alpha^2 x) L_{N-1/2 \pm 1/2}^1(\alpha^2 x) e^{-(1+\alpha^2)x}, \end{aligned} \quad (3.20)$$

where  $\alpha = d/l$  and  $L_N^m(x)$  is associated Laguerre's polynomial. Corresponding expression for  $\Gamma_N^2$  is given by eqs. (2.24) and (2.26) in ref. 7. For the ground Landau level, for example, one gets

$$\left( \frac{\Gamma_0^{xy}}{\Gamma_0} \right)^4 = \frac{1}{2(1+\alpha^2)^2}. \quad (3.21)$$

In Fig. 3,  $(\Gamma_N^{xy}/\Gamma_N)^4/(N+1/2)$  is plotted against the range  $\alpha$  for several Landau levels. Therefore,

$\Delta\sigma_{XY}^{(2)}$  decreases very rapidly with increasing range if  $\Gamma_N$  is kept constant.

### 3.2 Short-ranged scatterers under magnetic fields of arbitrary strength

In this section we specialize ourselves to the case of short-ranged scatterers. By the use of eqs. (3.11) and (3.12), eq. (3.3) becomes

$$\xi_{N\pm 1N}^x(E-i0, E+i0) = \pm 2i\sqrt{\frac{N+1/2 \pm 1/2}{2}} \text{Im} \Sigma(E+i0), \quad (3.22)$$

where

$$\Sigma(E) = \frac{1}{4} \Gamma^2 \sum_N G_N(E). \quad (3.23)$$

Since the vertex part  $\gamma_{N_1 \pm 1N_1, N_2 \pm 1N_2}$  vanishes, the Hall conductivity becomes

$$\begin{aligned} \Delta\sigma_{XY} &= \frac{e^2}{2\pi^2\hbar} \int \left( -\frac{\partial f}{\partial E} \right) dE 2(\text{Im} \Sigma(E+i0))^2 \sum_N (N+1) \text{Im} [G_N(E+i0)G_{N+1}(E-i0)] \\ &= \frac{e^2}{2\pi^2\hbar} \int \left( -\frac{\partial f}{\partial E} \right) dE \frac{-2 \text{Im} \Sigma(E+i0)}{\hbar\omega_c} \sum_N (N+1)(\hbar\omega_c)^2 \text{Im} G_N(E+i0) \text{Im} G_{N+1}(E+i0). \end{aligned} \quad (3.24)$$

On the other hand, the transverse conductivity  $\sigma_{XX}$  is written as

$$\sigma_{XX} = \frac{e^2}{2\pi^2\hbar} \int \left( -\frac{\partial f}{\partial E} \right) dE \sum_N (N+1)(\hbar\omega_c)^2 \text{Im} G_N(E+i0) \text{Im} G_{N+1}(E+i0). \quad (3.25)$$

One notices that at zero temperature

$$\Delta\sigma_{XY} = \frac{-2 \text{Im} \Sigma(E_F+i0)}{\hbar\omega_c} \sigma_{XX}, \quad (3.26)$$

where  $E_F$  is the Fermi energy. If one puts  $\text{Im} \Sigma(E_F+i0) = -\hbar/2\tau$ , eq. (3.26) is the same as the phenomenological formula (1.3).\*

Let us consider the case that the Fermi level lies in high Landau levels at zero temperature. The discussion in IV applies directly to the present  $\sigma_{XY}$ . When the magnetic field is sufficiently strong ( $\omega_c\tau_f \gg 1$ ), one gets.

$$\begin{aligned} \Delta\sigma_{XY} &= \frac{e^2}{\pi^2\hbar} \left( N + \frac{1}{2} \right) \frac{\Gamma}{\hbar\omega_c} \\ &\times \left[ 1 + \left( \frac{\pi^2}{8} - 1 \right) \left( \frac{\Gamma}{\hbar\omega_c} \right)^2 + \dots \right], \end{aligned} \quad (3.27)$$

or

$$\frac{\Delta\sigma_{XY}}{\sigma_{XX}} = \frac{\Gamma}{\hbar\omega_c} \left[ 1 + \frac{\pi^2}{24} \left( \frac{\Gamma}{\hbar\omega_c} \right)^2 + \dots \right], \quad (3.28)$$

at each peak associated with the  $N$ -th Landau level. When the magnetic field is rather weak ( $\omega_c\tau_f \lesssim 1$ ), on the other hand, one has

$$\sigma_{XY} = -\frac{ne^2\tau_f}{m} \frac{\omega_c\tau_f}{1 + \omega_c^2\tau_f^2}$$

\* As is easily seen, eq. (3.26) is applicable to three-dimensional systems. Further it is easy to show that eq. (3.26) still holds for  $\Delta\sigma_{XY}^{(2)}$  even if effects of higher Born scattering are included within the single-site approximation (SSA).

$$\begin{aligned} &\times \left[ 1 - \frac{2(1+3\omega_c^2\tau_f^2)}{\omega_c^2\tau_f^2(1+\omega_c^2\tau_f^2)} \cos \frac{2\pi X'}{\hbar\omega_c} \right. \\ &\times \exp \left( -\frac{\pi}{\omega_c\tau_f} \right) \left. \right], \end{aligned} \quad (3.29)$$

where  $X'$  is defined in IV.

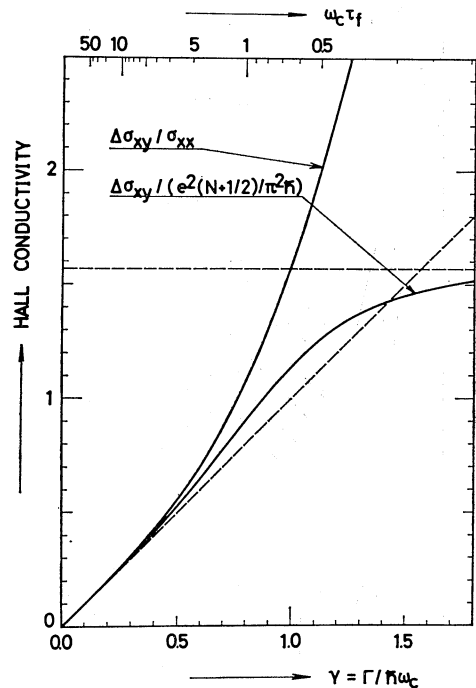


Fig. 4.  $\Delta\sigma_{XY}$  and  $\Delta\sigma_{XY}/\sigma_{XX}$  as a function of  $\gamma = \Gamma/\hbar\omega_c$  at  $X' = N\hbar\omega_c$ . See also Fig. 2 in IV.

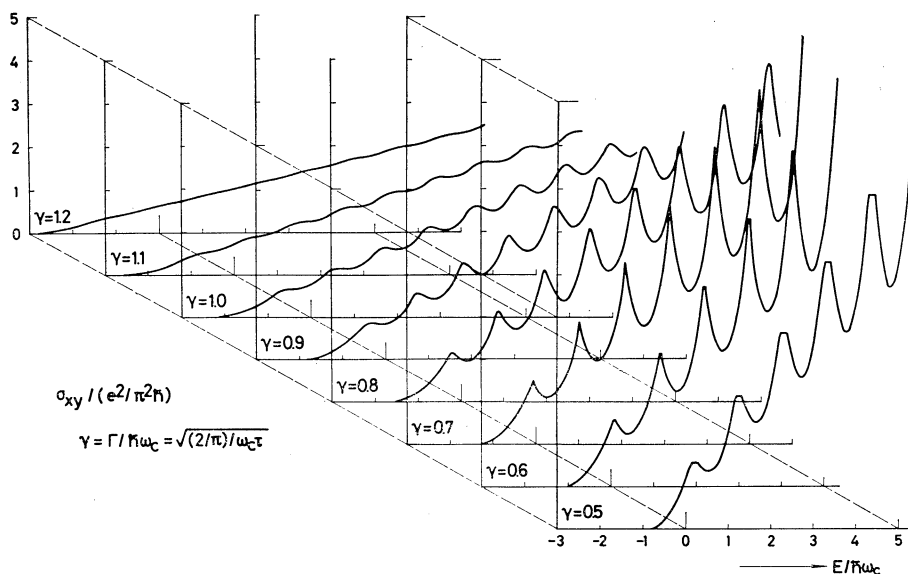


Fig. 5. Examples of oscillatory  $\sigma_{xy}$  for several  $\gamma = \Gamma / \hbar \omega_c$ .

In Fig. 4,  $\Delta\sigma_{xy}$  and  $\Delta\sigma_{xy}/\sigma_{xx}$  at  $X' = N\hbar\omega_c$  for general  $\gamma = \Gamma/\hbar\omega_c$ , which are numerically calculated, are shown as a function of  $\gamma$ . The oscillatory conductivity  $\sigma_{xy}$  calculated numerically in a similar manner to IV is given in Fig. 5.

#### §4. Higher Approximations under Strong Magnetic Fields

First, we investigate effects of higher order scattering from a single scatterer employing the single-site approximation (SSA) as in II. When the magnetic field is sufficiently strong, one should include diagrams corresponding to those shown in Fig. 2 of II and one gets

$$\begin{aligned} \Delta\sigma_{xy}^{(1)} = & \frac{e^2}{\pi^2 \hbar} \int f(E) dE 2\pi l^2 \sum_{\mu} \int dZ N_i^{(\mu)}(Z) \sum_m \sum_{\pm} \frac{1}{2i} \left[ \left( Nm \pm 1 \left| l \frac{\partial v^{(\mu)}(Z)}{\partial y} \right| Nm \right) \right. \\ & \times \left( Nm \left| l \frac{\partial v^{(\mu)}(Z)}{\partial x} \right| Nm \pm 1 \right) - (x \leftrightarrow y) \left. \right] \\ & \times \text{Im} \frac{1}{X_N - v_{Nm}^{(\mu)}(Z)} \frac{\partial}{\partial E} \text{Re} \frac{1}{X_N - v_{Nm \pm 1}^{(\mu)}(Z)}, \end{aligned} \quad (4.1)$$

where  $X_N(E) = G_N(E)^{-1}$ ,  $v_{Nm}^{(\mu)}(Z) = (Nm | v^{(\mu)}(Z) | Nm)$ , and one should choose  $N$  in such a way that  $E_N$  is closest to  $E$ . Making use of eq. (2.8) and integrating by parts, one has

$$\begin{aligned} \Delta\sigma_{xy}^{(1)} = & \frac{e^2}{\pi^2 \hbar} \int \left( -\frac{\partial f}{\partial E} \right) dE 2\pi l^2 \sum_{\mu} \int dZ N_i^{(\mu)}(Z) \sum_m \left| \left( Nm \left| l \frac{\partial v^{(\mu)}(Z)}{\partial x} \right| Nm + 1 \right) \right|^2 \\ & \times \left\{ [v_{Nm}^{(\mu)}(Z) - v_{Nm+1}^{(\mu)}(Z)]^{-2} \text{Im} \left[ \ln \frac{X_N}{X_N - v_{Nm}^{(\mu)}(Z)} - \frac{v_{Nm}^{(\mu)}(Z)}{X_N - v_{Nm}^{(\mu)}(Z)} \right. \right. \\ & \left. \left. - \ln \frac{X_N}{X_N - v_{Nm+1}^{(\mu)}(Z)} + \frac{v_{Nm+1}^{(\mu)}(Z)}{X_N - v_{Nm+1}^{(\mu)}(Z)} \right] \right. \\ & \left. + [v_{Nm}^{(\mu)}(Z) - v_{Nm+1}^{(\mu)}(Z)]^{-1} \left[ \frac{v_{Nm}^{(\mu)}(Z)}{X_N - v_{Nm}^{(\mu)}(Z)} \text{Im} \frac{1}{X_N - v_{Nm}^{(\mu)}(Z)} \right. \right. \\ & \left. \left. \times \text{Re} \frac{1}{X_N - v_{Nm+1}^{(\mu)}(Z)} + v_{Nm+1}^{(\mu)}(Z) \text{Re} \frac{1}{X_N - v_{Nm}^{(\mu)}(Z)} \text{Im} \frac{1}{X_N - v_{Nm+1}^{(\mu)}(Z)} \right] \right\}, \end{aligned} \quad (4.2)$$

where such branch of logarithm should be chosen whose imaginary part lies between  $-\pi$  and  $\pi$ . With the aid of eq. (2.9) of II and the relation

$$\left| \left( Nm \left| l \frac{\partial v^{(\mu)}(Z)}{\partial x} \right| Nm+1 \right) \right|^2 = \frac{1}{2} (N+m+1) [(Nm+1|v^{(\mu)}(Z)|Nm+1) - (Nm|v^{(\mu)}(Z)|Nm)]^2, \quad (4.3)$$

which is given in Appendix of II, the Hall conductivity becomes

$$\begin{aligned} \sigma_{xy} &= -\frac{nec}{H} + \Delta\sigma_{xy}^{(1)}, \\ &= -\frac{e^2}{\pi^2\hbar} \int \left( -\frac{\partial f}{\partial E} \right) dE \left\{ \frac{\pi}{2} \left[ N+1 - \frac{1}{\pi} \text{Im} \ln X_N \right] \right. \\ &\quad - 2\pi l^2 \sum_{\mu} \int dZ N_i^{(\mu)}(Z) \sum_m \left| \left( Nm \left| l \frac{\partial v^{(\mu)}(Z)}{\partial x} \right| Nm+1 \right) \right|^2 \\ &\quad \times [v_{Nm}^{(\mu)}(Z) - v_{Nm+1}^{(\mu)}(Z)]^{-1} \left[ v_{Nm}^{(\mu)}(Z) \text{Im} \frac{1}{X_N - v_{Nm}^{(\mu)}(Z)} \text{Re} \frac{1}{X_N - v_{Nm+1}^{(\mu)}(Z)} \right. \\ &\quad \left. \left. + v_{Nm+1}^{(\mu)}(Z) \text{Re} \frac{1}{X_N - v_{Nm}^{(\mu)}(Z)} \text{Im} \frac{1}{X_N - v_{Nm+1}^{(\mu)}(Z)} \right] \right\}. \end{aligned} \quad (4.4)$$

The Green's function  $G_N$  or  $X_N$  is determined by the self-consistency equation (2.8) of II. From the above equation, one can conclude that  $\Delta\sigma_{xy}^{(1)}$  vanishes and  $\sigma_{xy}$  becomes  $-nec/H = -e^2(N+1)/2\pi\hbar$ , when the Fermi level lies in energy gaps between adjacent  $N$ -th and  $N+1$ -th Landau levels at zero temperature.

Let us consider the case that concentrations of scatterers are sufficiently small and impurity bands are separated from the  $N$ -th Landau level. At the energy where the spectrum has a gap between two impurity bands or between an impurity band and the main Landau level,  $X_N$  becomes a negative real number in case of attractive scatterers and a positive one in case of repulsive scatterers. When the Fermi level lies in those spectral gaps at zero temperature, therefore, the Hall conductivity becomes  $-e^2N/2\pi\hbar$  in case of attractive scatterers and  $-e^2(N+1)/2\pi\hbar$  in case of repulsive scatterers.\* This means that electrons which fully occupy impurity bands do not contribute to the Hall current, while those which occupy the main Landau level give rise to the same Hall current as that obtained when all i.e.  $1/2\pi l^2$  electrons of the Landau level move freely.

In case of high concentrations of scatterers, the absolute value of  $X_N$  becomes large compared with  $(Nm|v^{(\mu)}(Z)|Nm)$  and further  $X_N$  approaches a solution in the SCBA

$$X_N = \frac{1}{2} [E - E_N + i\sqrt{\Gamma_N'^2 - (E - E_N)^2}], \quad (4.5)$$

Therefore, one has

$$\Delta\sigma_{xy}^{(1)} = -\frac{e^2}{3\pi^2\hbar} \int \left( -\frac{\partial f}{\partial E} \right) dE \left( \frac{\Gamma_N'}{\Gamma_N} \right)^3 \left[ 1 - \left( \frac{E - E_N}{\Gamma_N} \right)^2 \right]^{3/2}, \quad (4.6)$$

with

$$\begin{aligned} (\Gamma_N')^3 &= 2^3 \cdot (2\pi l^2) \sum_{\mu} \int dZ N_i^{(\mu)}(Z) \sum_m \left| \left( Nm \left| l \frac{\partial v^{(\mu)}(Z)}{\partial x} \right| Nm+1 \right) \right|^2 \\ &\quad \times [(Nm|v^{(\mu)}(Z)|Nm) - (Nm+1|v^{(\mu)}(Z)|Nm+1)]. \end{aligned} \quad (4.7)$$

Especially in case of short-ranged scatterers, eq. (4.7) becomes

$$(\Gamma_N')^3 = 2^3 \cdot 2\pi l^2 \sum_{\mu} \int dZ N_i^{(\mu)}(Z) \left[ \frac{V^{(\mu)}(Z)}{2\pi l^2} \right]^3. \quad (4.8)$$

As can be seen from the above equations, the effects of higher Born scattering does not vanish in the limit of  $\Gamma/\hbar\omega_c \rightarrow 0$ , and further they depend on the sign of scatterers:  $\Delta\sigma_{xy}^{(1)}$  becomes positive (negative) when scatterers are attractive (repulsive). It should be noticed also that  $\Delta\sigma_{xy}^{(1)}$  does

\* A repulsive scatterer can have impurity bands in our system.



not increase with the Landau level  $N$  in contrast to  $\Delta\sigma_{xy}^{(2)}$  and that it is independent of  $N$  in case of short-ranged scatterers. With the increase of the concentration of scatterers,  $\Delta\sigma_{xy}^{(1)}$  becomes small, which is consistent with the result obtained in II that the SSA approaches the SCBA in case of high concentrations of scatterers.

As an illustrative example, we consider a model system in which a single kind of scatterers with the Gaussian potential (3.19) is distributed randomly. In a similar manner to in § 3 of II, one can calculate  $\sigma_{xy}$  of the ground Landau level. Examples of the results are shown in Fig. 6.

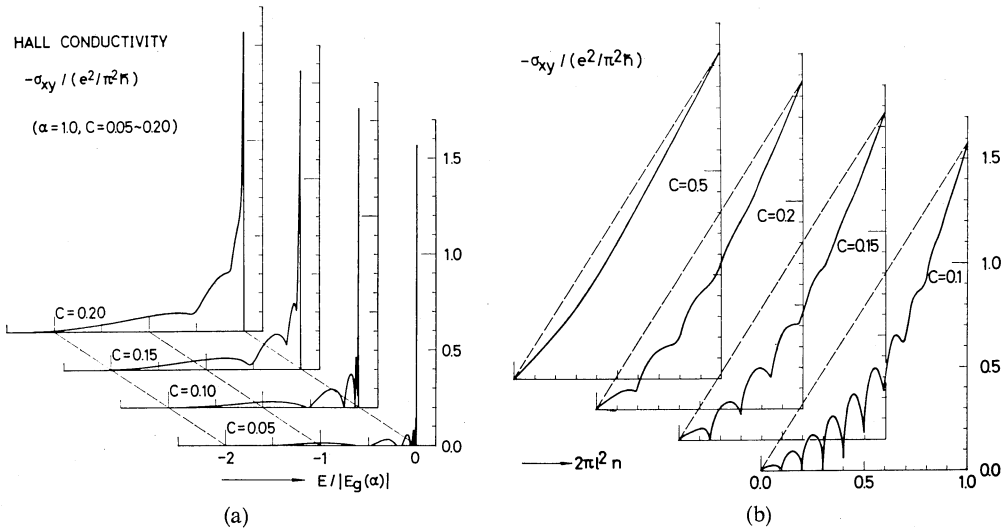


Fig. 6. Examples of  $\sigma_{xy}$  of the ground Landau level in case of scatterers with a Gaussian potential.  $c = 2\pi l^2 N_i$ . See also Figs. 6~11. (a)  $\sigma_{xy}$  as a function of the energy. (b)  $\sigma_{xy}$  as a function of total number of electrons.

Next we investigate effects of simultaneous scattering from more than two scatterers to the lowest order, confining ourselves to the case of sufficiently high concentrations of short-ranged scatterers. One has to take into account those diagrams shown in Fig. 1(b) and gets

$$\Delta\sigma_{xy}^{(1)} = \frac{e^2}{\pi^2 \hbar} \frac{\Gamma^4}{4^3} \int f(E) dE h_N^{(2)} \left[ \text{Im } G_N^3 \frac{\partial}{\partial E} \text{Re } G_N - \text{Im } G_N \frac{\partial}{\partial E} \text{Re } G_N^3 \right], \quad (4.9)$$

with

$$h_N^{(2)} = 4 \cdot \int \frac{d^2 R}{2\pi l^2} \left\{ \frac{N+1}{2} [J_{N+1N+1}(R) J_{NN}(R) + J_{N+1N}(R)^2] - \frac{N}{2} [J_{NN}(R) J_{N-1N-1}(R) + J_{NN-1}(R)^2] \right\}. \quad (4.10)$$

Partial integration of eq. (4.9) yields

$$\Delta\sigma_{xy}^{(1)} = \frac{e^2}{\pi^2 \hbar} \frac{\Gamma^4}{4^3} \int \left( -\frac{\partial f}{\partial E} \right) dE h_N^{(2)} \left[ \frac{1}{2} \text{Im } G_N^2 |G_N|^2 - \frac{1}{4} \text{Im } G_N^4 \right]. \quad (4.11)$$

If one uses  $G_N$  given by eq. (4.5) of the SCBA, it becomes

$$\Delta\sigma_{xy}^{(1)} = -\frac{e^2}{\pi^2 \hbar} \frac{1}{2} h_N^{(2)} \int \left( -\frac{\partial f}{\partial E} \right) dE \frac{E - E_N}{\Gamma} \left[ 1 - \left( \frac{E - E_N}{\Gamma} \right)^2 \right]^{3/2}, \quad (4.12)$$

the integrand of which becomes zero at  $E = E_N$  and becomes large at tails of the spectrum. One has, for example,  $h_0^{(2)} = 1$ ,  $h_1^{(2)} = 3/4, \dots$ , and  $h_N^{(2)}$  decreases with the increase of  $N$ . Therefore, many-site corrections are not im-

portant for large Landau level index  $N$ .

## § 5. Discussion and Conclusion

The relation (3.26) between  $\Delta\sigma_{xy}$  and  $\sigma_{xx}$  in the SCBA can physically be understood as fol-

lows. When an electric field  $E_y$  is applied in the  $y$ -direction, a center of the cyclotron motion of electrons moves in the  $x$ -direction with a drift velocity  $v_x = cE_y/H$ , which gives rise to the Hall conductivity  $-e\mu/H$ . Effects of scattering can be regarded as a frictional force acting on each electron, the strength of which is given by  $F_x = -mv_x/\tau = -eE_y/\omega_c\tau$ , where  $\tau$  is a relaxation time. The current in the  $x$ -direction due to this force becomes

$$\Delta j_x = \sigma_{xx} \frac{F_x}{(-e)} = \frac{1}{\omega_c\tau} \sigma_{xx} E_y = \Delta\sigma_{xy} E_y. \quad (5.1)$$

Therefore, one has eq. (3.26).

MOS inversion layers on Si (100) surface are a typical two-dimensional system. For usual electron concentrations ( $N_{inv} \gtrsim 10^{12} \text{ cm}^{-2}$ ), main scatterers of this system are considered to be of short range and their concentrations are considered to be relatively large. Therefore, the results in the SCBA together with eq. (4.6) are expected to apply to inversion layers. Measurements of both  $\sigma_{xx}$  and  $\sigma_{xy}$  will give useful informations on main scatterers and especially on the level broadening  $\Gamma$ . If one compares line-shapes obtained from eq. (3.6) of I and eqs. (3.13) and (4.6), one sees that the effective width of  $\Delta\sigma_{xy}$  is smaller than that of  $\sigma_{xx}$ . Therefore, spin and valley splittings are expected to be seen in  $\sigma_{xy}$  at higher Landau levels than in  $\sigma_{xx}$ . Recently Igarashi, Wakabayashi and Kawaji measured these quantities.<sup>8)</sup> The overall

behavior of their results is satisfactorily in agreement with the present theory. Detailed comparison will be made in their paper.

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